

# Gravitational domain walls and $p$ -brane distributions <sup>\*</sup>

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## Abstract

We review the main algebraic aspects that characterize and determine the domain wall solutions of maximal gauged supergravity in various spacetime dimensions by considering consistent truncations that retain a number of components in the diagonal of the coset space scalar manifold of the theory. Starting from the algebraic classification of domain walls in  $D = 4$  gauged supergravity, we also derive the corresponding solutions in  $D = 5$  and  $D = 7$  dimensions. From a higher dimensional point of view, these solutions describe the gravitational field, in the field theory limit, of a large number of M2-, D3- and M5-branes that are distributed on hypersurfaces in the transverse space to the branes. As a new result we employ a smearing procedure as well as various dualities to list the irreducible curves and the symmetry groups of  $p$ -brane distributions for all values of  $p$  that are of interest in current applications of string theory. Some emphasis is placed on the presentation of new results in the case of NS5-branes.

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# 1 Gravitational domain walls

It is well known that gravitational theories can admit domain wall solutions in the presence of scalar fields. In particular,  $D$ -dimensional gravity coupled to a scalar field  $\phi$  with potential  $V(\phi)$  that is derived from a superpotential  $W(\phi)$ , as in

$$S[g, \phi] = \int d^D x \left( \frac{1}{4} R - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right), \quad (1.1)$$

with

$$V(\phi) = \frac{g^2}{8} \left( \left( \frac{\partial W}{\partial \phi} \right)^2 - 2 \frac{D-1}{D-2} W^2 \right), \quad (1.2)$$

serves as a prototype for constructing stable solutions that depend on a single spatial coordinate, say  $r$ , using the special ansatz for the spacetime metric

$$ds^2 = dr^2 + e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu; \quad \mu, \nu = 0, 1, \dots, D-2, \quad (1.3)$$

which preserves Poincaré invariance in  $D-1$  dimensions. The domain walls satisfy the first order equations

$$\frac{d\phi}{dr} = \mp \frac{g}{2} \frac{\partial W}{\partial \phi}, \quad \frac{dA}{dr} = \pm \frac{g}{D-2} W, \quad (1.4)$$

which are equivalent conditions for having 1/2-BPS supersymmetric configurations in the bosonic sector of gauged supergravity, in appropriate generalizations of the model.

Extending the standard Bogomol'nyi argument to the gravitational case, one is lead to consider the following effective functional

$$E[A, \phi] = \int_{-\infty}^{+\infty} dr e^{(D-1)A} \left( \frac{1}{2} (\partial_r \phi)^2 + V(\phi) - \frac{1}{4} (D-1)(D-2) (\partial_r A)^2 \right), \quad (1.5)$$

whose extrema provide an equivalent description of all solutions of the theory (1.1) in the sector with metric form (1.3). The special set of first order equations (1.4) that characterize the domain walls follow easily by completing the squares in the integrant of the effective functional as

$$\begin{aligned} E &= \frac{1}{2} \int_{-\infty}^{+\infty} dr e^{(D-1)A} \left( \left( \partial_r \phi \pm \frac{g}{2} \frac{\partial W}{\partial \phi} \right)^2 - \frac{1}{2} (D-1)(D-2) \left( \partial_r A \mp \frac{g}{D-2} W \right)^2 \right) \\ &\mp \frac{g}{2} W e^{(D-1)A} \Big|_{-\infty}^{+\infty}, \end{aligned} \quad (1.6)$$

where  $\pm\infty$  represent the end points in the range of the variable  $r$ . When each one of the two perfect square terms vanishes separately, the effective functional receives contribution only from the boundary term, and the resulting first order equations arise as its saddle points (accounting for the relative  $-$  sign between the two terms); as such they also provide solutions of the full set of second order equations.

We note for completeness that the gravitational domain walls provide a natural generalization of the usual kink solutions arising in the 2-dim scalar field theory

$$S[\phi] = \int d^2 x \left( -\frac{1}{2} (\partial\phi)^2 - V(\phi) \right); \quad V(\phi) = \frac{g^2}{8} \left( \frac{\partial W}{\partial \phi} \right)^2, \quad (1.7)$$

which are obtained by minimizing the energy functional of its static configurations a la Bogomol'nyi

$$E[\phi] = \int_{-\infty}^{+\infty} dx \left( \frac{1}{2} (\partial_x \phi)^2 + V(\phi) \right) = \frac{1}{2} \int_{-\infty}^{+\infty} dx \left( \partial_x \phi \pm \frac{g}{2} \frac{\partial W}{\partial \phi} \right)^2 \mp \frac{g}{2} W|_{-\infty}^{+\infty} . \quad (1.8)$$

In fact, it can be seen that (1.7) and (1.8) follow from the corresponding equations in the gravitational case, but in the limit where gravity decouples. Indeed, in (1.1) above we have set the gravitational coupling constant  $\kappa_D = \sqrt{2}$ , but if we reinstate it and perform a rescaling of the scalar fields, gravity will decouple in the limit  $\kappa_D \rightarrow 0$ , as stated.

Some basic facts, as well as various applications of gravitational domain walls, can be found in the review [1] (and references therein). The results we present here are based on earlier work in [2]; also some closely related results can be found in [3] (and references therein).

## 2 Solutions of gauged supergravity

In order to construct domain wall solutions of gauged supergravity in various dimensions, it is first convenient to extend the formalism to a multi scalar field context with diagonal target space metric, namely

$$S[g, \phi_I] = \int d^D x \left( \frac{1}{4} R - \frac{1}{2} \sum_{I=1}^{N-1} (\partial \phi_I)^2 - \frac{g^2}{8} \left( \sum_{I=1}^{N-1} \left( \frac{\partial W}{\partial \phi_I} \right)^2 - 2 \frac{D-1}{D-2} W^2 \right) \right) . \quad (2.1)$$

Then, the corresponding system of first order equations turns out to be

$$\frac{d\phi_I}{dr} = \frac{g}{2} \frac{\partial W}{\partial \phi_I} , \quad \frac{dA}{dr} = -\frac{g}{D-2} W , \quad (2.2)$$

where one of the two signs has been chosen for simplicity and  $g$  will be set equal to 1 from now on, unless otherwise stated.

We consider the sector of maximally gauged supergravity in various dimensions in which the scalar field manifold assumes the coset space form  $SL(N, \mathbb{R})/SO(N)$ . A further consistent truncation is to confine ourselves to the diagonal components, setting all other scalar and gauge fields equal to zero. As we will see shortly, this particular sector turns out to be exactly solvable by employing techniques of algebraic geometry. It is convenient to introduce  $N$  scalar fields  $\beta_i$  subject to the constraint

$$\beta_1 + \beta_2 + \cdots + \beta_N = 0 , \quad (2.3)$$

which are related to the original fields  $\phi_I$  as in

$$\beta_i = \sum_{I=1}^{N-1} \lambda_{iI} \phi_I , \quad (2.4)$$

with  $\lambda_{iI}$  being the fundamental weights of  $SL(N)$  algebra, which satisfy the defining relations

$$\sum_{i=1}^N \lambda_{iI} = 0, \quad \sum_{i=1}^N \lambda_{iI} \lambda_{iJ} = 2\delta_{IJ}, \quad \sum_{I=1}^{N-1} \lambda_{iI} \lambda_{jI} = 2\delta_{ij} - \frac{2}{N}. \quad (2.5)$$

Using this data, the superpotential of gauged supergravity assumes the simple form

$$W = -\frac{1}{4} \sum_{i=1}^N e^{2\beta_i}. \quad (2.6)$$

We will construct domain wall equations of the theory using the conformally flat form of the metric

$$ds^2 = e^{2A(z)} (dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu), \quad (2.7)$$

or else the coordinate  $z$ , instead of  $r$ , related as  $(\exp A)dz = -dr$ . Writing the first order equations for the domain walls, we obtain the following system in terms of the set of fields  $\beta_i$ :

$$\beta'_i = \frac{1}{2} e^{2\beta_i + A} + 2 \frac{D-2}{N} A', \quad A' = \frac{1}{D-2} e^A W, \quad (2.8)$$

with  $i = 1, 2, \dots, N$ , and prime denotes the derivative with respect to the variable  $z$ . These equations are easy to integrate, at least formally at the moment, by writing the conformal factor of the metric as

$$e^{A(z)} = (-F'(z))^{\frac{N}{N+4(D-2)}}, \quad (2.9)$$

in terms of some (yet unknown) function  $F(z)$ . Then, one easily finds that the scalar fields are given by

$$e^{2\beta_i(z)} = \frac{(-F'(z))^{\Delta/N}}{F(z) - b_i}, \quad (2.10)$$

where  $b_i$  are some integration constants (moduli) for all  $i = 1, 2, \dots, N$  and the exponent is  $\Delta = 4N(D-2)/(N+4(D-2))$ . Thus, according to this substitution, all it remains open is the differential equation for finding the suitable function  $F(z)$ . This can be determined by appealing to the constraint (2.3) among the fields  $\beta_i$ . Using their exponential form above, we arrive by multiplication to the non-linear equation

$$(-F'(z))^\Delta = \prod_{i=1}^N (F(z) - b_i) \equiv f, \quad (2.11)$$

which has to be solved in all cases of interest.

The method we developed so far is quite general. For application to theories of gauged supergravity we confine ourselves to the following cases for the dimensionality of the spacetime  $D$  and the number of scalar fields  $N$ :  $(D, N) = (4, 8)$ ,  $(5, 6)$  and  $(7, 5)$ . Note that in all three case the exponent of the corresponding non-linear differential equation is  $\Delta = 4$ . Of course, appropriate boundary conditions will be introduced later in the integration of the master equation (2.11). Also, the range of  $z$  has to be such that  $F(z) \geq b_{\max}$  (the maximum value of the moduli  $b_i$ ) in order for the scalar fields  $\beta_i$  to assume real values.

Note at this point that had we chosen to work with the form of the metric (1.3), using the variable  $r$  instead of  $z$ , a similar ansatz would work for expressing the domain wall solutions in terms of an unknown function  $F(r)$ , namely

$$e^{A(r)} = \left( \dot{F}(r) \right)^{\frac{N}{4(D-2)}} , \quad e^{2\beta_i(r)} = \frac{\dot{F}(r)}{F(r) - b_i} , \quad (2.12)$$

where dot denotes the derivative with respect to  $r$  and  $b_i$  denote the appropriate moduli of integration. In this case we easily find  $(-F'(z))^\Delta = (\dot{F}(r))^N$ , and so the corresponding non-linear differential equation for  $F(r)$  reads

$$\left( \dot{F}(r) \right)^N = \prod_{i=1}^N (F(r) - b_i) . \quad (2.13)$$

There are two advantages in using the variable  $z$  instead of  $r$ . First, the spectrum of the transverse traceless graviton fluctuations, obeying the massless scalar field equation,

$$\Phi(x, z) = \exp(ik \cdot x) \exp\left(-\frac{D-2}{2}A(z)\right) \Psi(z) , \quad (2.14)$$

which represent plane waves propagating along the  $(D-2)$ -brane with a  $z$ -dependent amplitude, can be cast directly into a time-independent Schrödinger problem

$$-\Psi''(z) + V(z)\Psi(z) = M^2\Psi(z) , \quad (2.15)$$

where  $M^2 = -k \cdot k$ . The corresponding Schrödinger potential assumes the form

$$V(z) = W^2(z) + W'(z) ; \quad W(z) \equiv \frac{D-2}{2}A'(z) \quad (2.16)$$

and hence enjoys all properties of supersymmetric quantum mechanics. Second, as we will see next, it is possible to treat the three cases  $(D, N)$  that are of interest in gauged supergravity all at once, because the value of the exponent  $\Delta = 4$  is universal. Then, starting from  $(D, N) = (4, 8)$  we will obtain results for the other cases too by a simple limiting procedure applied to the algebraic classification of the corresponding domain wall solutions.

### 3 Algebraic classification

The non-linear differential equation (2.11) with  $\Delta = 4$  can be viewed, when it is extended to the complex domain, as a Christoffel–Schwarz transformation from a closed polygon in the  $z$ -plane onto the upper-half  $F$ -plane. In this case the perimeter of the polygon is mapped to the real  $F$ -axis, whereas its vertices are mapped to points parametrized by the moduli  $b_i$ . It is convenient to start from the case  $(D, N) = (4, 8)$  by considering a closed octagon and the Christoffel–Schwarz transformation

$$\frac{dz}{dF} = (F - b_1)^{-\varphi_1/\pi} (F - b_2)^{-\varphi_2/\pi} \dots (F - b_8)^{-\varphi_8/\pi} , \quad (3.1)$$

making the canonical choice

$$\varphi_1 = \varphi_2 = \cdots = \varphi_8 = \frac{\pi}{4} , \quad (3.2)$$

for generic values of the moduli  $b_i$ . Letting

$$x = F(z) , \quad y = F'(z) , \quad (3.3)$$

we arrive at the algebraic curve

$$y^4 = (x - b_1)(x - b_2) \cdots (x - b_8) , \quad (3.4)$$

which has to be *uniformized* in terms of a complex variable, say  $u$ , as  $x = x(u)$  and  $y = y(u)$ . This is a difficult problem in practice for generic values of the moduli  $b_i$ , but as soon as this is done we may obtain  $u(z)$  by inverting the solution of the equation

$$\frac{dz}{du} = \frac{1}{y(u)} \frac{dx(u)}{du} \quad (3.5)$$

and hence construct the desired solution of the master equation that determines the domain walls of maximal gauged supergravity in four spacetime dimensions according to the ansatz (2.9) and (2.10).

It is clear that in the corner of the moduli space, where all  $b_i$  become equal to each other, the scalar fields vanish and the geometry of the metric coincides with that of  $AdS_4$ , as it should. In fact, this solution provides the boundary condition for integrating the equation (3.5), and hence determine the domain wall solutions for generic points of the moduli space, namely, we demand that the domain wall approaches the trivial  $AdS_4$  solution as  $F \rightarrow \infty$  (or equivalently letting  $z \rightarrow 0^+$ , which fixes the origin). Appropriate restrictions also have to be introduced so that the resulting solutions for the scalar fields and the conformal factor of the metric turn out to be real, despite the original formulation of the Christoffel–Schwarz transformation in the complex domain.

Standard techniques from algebraic geometry yield the classification of the algebraic curves in table 1 below (written in irreducible form), which describe the domain walls in  $D = 4$ ,  $N = 8$  according to genus. We also note that the case where the unbroken symmetry group equals the Cartan subgroup of  $SO(8)$ , i.e.  $SO(2)^4$ , it corresponds to the extremal supersymmetric limit of the most general rotating M2-brane solution which depends on four rotating parameters. In general, the same remark holds true for the tables corresponding to all  $p$ -branes that will be listed later in the paper. Namely, when the original symmetry group  $SO(N)$  is broken down to its Cartan subgroup, the corresponding solution is the extremal supersymmetric limit of the most general rotating  $p$ -brane solution, with the number of its rotating parameters equal to the dimension of the Cartan subgroup of  $SO(N)$ .

<i>Genus</i>	<i>Irreducible Curve</i>	<i>Isometry Group</i>
9	$y^4 = (x - b_1)(x - b_2) \cdots (x - b_7)(x - b_8)$	None
7	$y^4 = (x - b_1)(x - b_2) \cdots (x - b_6)(x - b_7)^2$	$SO(2)$
6	$y^4 = (x - b_1)(x - b_2) \cdots (x - b_5)(x - b_6)^3$	$SO(3)$
5	$y^4 = (x - b_1) \cdots (x - b_4)(x - b_5)^2(x - b_6)^2$	$SO(2) \times SO(2)$
4	$y^4 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)^2(x - b_5)^3$	$SO(2) \times SO(3)$
3	$y^4 = (x - b_1) \cdots (x - b_4)(x - b_5)^4$	$SO(4)$
	$y^4 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)^5$	$SO(5)$
	$y^4 = (x - b_1)(x - b_2)(x - b_3)^3(x - b_4)^3$	$SO(3) \times SO(3)$
	$y^4 = (x - b_1)(x - b_2)(x - b_3)^2(x - b_4)^2(x - b_5)^2$	$SO(2) \times SO(2) \times SO(2)$
2	$y^4 = (x - b_1)(x - b_2)^2(x - b_3)^2(x - b_4)^3$	$SO(2) \times SO(2) \times SO(3)$
1	$y^4 = (x - b_1)(x - b_2)(x - b_3)^6$	$SO(6)$
	$y^4 = (x - b_1)(x - b_2)(x - b_3)^2(x - b_4)^4$	$SO(2) \times SO(4)$
	$y^4 = (x - b_1)(x - b_2)^2(x - b_3)^5$	$SO(2) \times SO(5)$
	$y^4 = (x - b_1)^2(x - b_2)^3(x - b_3)^3$	$SO(2) \times SO(3) \times SO(3)$
	$y^2 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)$	$SO(2)^4$
0	$y^4 = (x - b_1)(x - b_2)^7$	$SO(7)$
	$y = (x - b)^2$	$SO(8)$ (Maximal)
	$y^2 = (x - b_1)(x - b_2)^3$	$SO(2) \times SO(6)$
	$y^4 = (x - b_1)(x - b_2)^3(x - b_3)^4$	$SO(3) \times SO(4)$
	$y^4 = (x - b_1)^3(x - b_2)^5$	$SO(3) \times SO(5)$
	$y = (x - b_1)(x - b_2)$	$SO(4) \times SO(4)$
	$y^2 = (x - b_1)(x - b_2)(x - b_3)^2$	$SO(2)^2 \times SO(4)$

Table 1: Curves and symmetry groups of domain walls in  $D = 4$ ,  $N = 8$  supergravity.

The symmetry groups refer to the special regions of the moduli space where some of the parameters  $b_i$  are allowed to coincide, thus lowering the genus of the corresponding Riemann surfaces, which in turn lead to simplifications in the domain wall solutions as some of the scalar fields are linearly related to others.

The algebraic classification of the domain walls in  $D = 5$  supergravity with  $N = 6$  follows immediately from the list above by considering only those solutions with  $SO(2)$  isometry and letting the two coalescing moduli tend to infinity. Then, the non-linear differential equation (2.11) with  $N = 8$  becomes (after appropriate rescaling) the corresponding equation with  $N = 6$ , which is appropriate for  $D = 5$  supergravity. In the Christoffel–Schwarz transformation this amounts to degenerating the closed octagon by letting two of its vertices coincide and mapping the resulting double vertex to infinity. Thus, the classification presented in table 2 follows immediately:

<i>Genus</i>	<i>Irreducible Curve</i>	<i>Isometry Group</i>
7	$y^4 = (x - b_1)(x - b_2) \cdots (x - b_5)(x - b_6)$	None
5	$y^4 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)(x - b_5)^2$	$SO(2)$
4	$y^4 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)^3$	$SO(3)$
3	$y^4 = (x - b_1)(x - b_2)(x - b_3)^2(x - b_4)^2$	$SO(2) \times SO(2)$
2	$y^4 = (x - b_1)(x - b_2)^2(x - b_3)^3$	$SO(2) \times SO(3)$
1	$y^4 = (x - b_1)(x - b_2)(x - b_3)^4$	$SO(4)$
	$y^4 = (x - b_1)(x - b_2)^5$	$SO(5)$
	$y^4 = (x - b_1)^3(x - b_2)^3$	$SO(3) \times SO(3)$
	$y^2 = (x - b_1)(x - b_2)(x - b_3)$	$SO(2)^3$
0	$y^2 = (x - b_1)(x - b_2)^2$	$SO(2) \times SO(4)$
	$y^2 = (x - b)^3$	$SO(6)$ (Maximal)

Table 2: Curves and symmetry groups of domain walls in  $D = 5$ ,  $N = 6$  supergravity.

Likewise, letting three vertices first coincide and then mapping them to infinity, which amounts to factoring out an  $SO(3)$  isometry, we obtain table 3 for the algebraic classification of the domain walls in  $D = 7$  supergravity with  $N = 5$ :

<i>Genus</i>	<i>Irreducible Curve</i>	<i>Isometry Group</i>
6	$y^4 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)(x - b_5)$	None
4	$y^4 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)^2$	$SO(2)$
3	$y^4 = (x - b_1)(x - b_2)(x - b_3)^3$	$SO(3)$
2	$y^4 = (x - b_1)(x - b_2)^2(x - b_3)^2$	$SO(2)^2$
1	$y^4 = (x - b_1)^2(x - b_2)^3$	$SO(2) \times SO(3)$
0	$y^4 = (x - b_1)(x - b_2)^4$	$SO(4)$
	$y^4 = (x - b)^5$	$SO(5)$ (Maximal)

Table 3: Curves and symmetry groups of domain walls in  $D = 7$ ,  $N = 5$  supergravity.

Two technical remarks are in order before proceeding further. First, suppose that a moduli  $b$  of the Christoffel–Schwarz transformation underlying (2.11) is taken to infinity. This will lead to elimination of the corresponding factor  $(F - b)^\gamma$  from the algebraic curve, where  $\gamma$  denotes the associated degree of degeneracy of the vertex. It is practically achieved by first rescaling  $z$  to a new variable  $z' = z(-b)^{\gamma/\Delta}$  and then letting  $-b \rightarrow \infty$ . This actually amounts to rescaling the coupling constant  $g^2$  by a factor  $(-b)^{\gamma/\Delta}$ , which was previously set equal to 1 for simplicity, but it can be reinstated any moment in the various equations. We will see later that this limit provides a smearing for the various brane distributions which allows to construct, using also various  $U$ -duality transformations, solutions representing  $p$ -brane distributions for all values of  $p$  in string theory. For



the moment, it is sufficient to justify the classification presented in tables 2 and 3 following the complete list of table 1. We note, however, that in all three cases of interest in gauged supergravity the associated polygon in the  $z$  plane is closed for generic values of the moduli  $b_i$ ; it has 8, 7 and 6 vertices for  $(D, N) = (4, 8)$ ,  $(5, 6)$  and  $(7, 5)$ , respectively. Of course, at certain corners of the moduli space, where it degenerates further, the polygon may turn open for appropriate large isometry groups (see, for instance, table 1 for solutions with  $SO(n \geq 4)$  factors).

Second, the interpretation of the differential equation (2.13) as a Christoffel–Schwarz transformation in terms of the variable  $r$  differs from the interpretation given to equation (2.11) in terms of the variable  $z$  in that we have to consider the mapping of an open polygon in the  $r$  plane onto the upper-half  $F$ -plane. Namely, instead of an octagon at generic points, what we are considering now is an open polygon with  $N$  vertices each one having  $\pi/N$  as defect angle, and another vertex pulled at infinity in the  $r$ -plane, which necessarily has defect angle  $\pi$ ; furthermore, the latter is mapped to infinity in the  $F$ -plane. Therefore, changing variables from  $r$  to  $z$  is expected to be transcendental at generic points of the moduli space; put differently, the genus of the algebraic curves based on (2.11) or (2.13) will not be the same at generic points of the moduli space. It should be realized that this is not a problem as in the corresponding uniformization of the surfaces there are different multiple coverings along the branch cuts. At certain degenerate limits, however, where the isometry group of the solutions is appropriately chosen, the bounded regions in the  $z$ - or  $r$ -plane may have the same shape and hence the genus of the corresponding curves will be equal. Presently, we have chosen to work with the  $z$ -parametrization instead of  $r$  for the two main reasons that were explained in section 2.

## 4 Distributions of M2-, M5- and D3-branes

It was shown in the case of maximally gauged supergravity in  $D = 7, 4$  and 5 dimensions that the construction of domain walls gives rise to particular solutions of supergravity in higher dimensions, which describe the field theory limit of a large number of M5-, M2- and D3-branes distributed in various hypersurfaces embedded in the  $N$ -dimensional space transverse to the branes. In particular, the higher dimensional metrics for the various distributions of branes have the form

$$\text{M5-brane :} \quad ds^2 = H_0^{-1/3} \eta_{\mu\nu} dx^\mu dx^\nu + H_0^{2/3} (dy_1^2 + dy_2^2 + \dots + dy_5^2) , \quad (4.1)$$

$$\text{M2-brane :} \quad ds^2 = H_0^{-2/3} \eta_{\mu\nu} dx^\mu dx^\nu + H_0^{1/3} (dy_1^2 + dy_2^2 + \dots + dy_8^2) , \quad (4.2)$$

and

$$\text{D3-brane :} \quad ds^2 = H_0^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + H_0^{1/2} (dy_1^2 + dy_2^2 + \dots + dy_6^2) . \quad (4.3)$$

In all cases  $H_0$  is a harmonic function in the  $N$ -dimensional space  $\mathbb{R}^N$  transverse to the brane parametrized by the coordinates  $y_i$  and it is given by

$$H_0^{-1} = \frac{4}{R^4} f^{1/2} \sum_{i=1}^N \frac{y_i^2}{(F - b_i)^2} , \quad (4.4)$$

where  $f$  is defined in (2.11). The coordinate  $F$  is determined in terms of the transverse coordinates  $y_i$  as a solution of the algebraic equation

$$\sum_{i=1}^N \frac{y_i^2}{F - b_i} = 4g^{D-5} . \quad (4.5)$$

The algebraic equation (4.5) for  $F$  cannot be solved analytically for general choices of the constants  $b_i$ . However, this becomes practically possible when some of the  $b_i$ 's coincide in such a way that the degree of the corresponding algebraic equations with respect to  $F$  is reduced to 4 or less. It can also be shown that  $H_0$ , as defined in (4.4) and (4.5), is a harmonic function in  $\mathbb{R}^N$ , as it should. We may solve this constraint by introducing the change of coordinates

$$y_i = 2g^{(D-5)/2} (F - b_i)^{1/2} \hat{x}_i , \quad i = 1, 2, \dots, N , \quad (4.6)$$

where the  $\hat{x}_i$ 's define a unit  $N$ -sphere. Then, the  $N$ -dimensional flat metric in the transverse part of the brane metric (4.1)–(4.3) can be written as

$$\sum_{i=1}^N dy_i^2 = g^{D-5} \sum_{i=1}^N \frac{\hat{x}_i^2}{F - b_i} dF^2 + 4g^{D-5} \sum_{i=1}^N (F - b_i) d\hat{x}_i^2 . \quad (4.7)$$

The metrics (4.1)–(4.3) become asymptotically  $AdS_D \times S^{N-1}$  for large radial distances, with  $D$  and  $N$  taking their appropriate values. The radius of the sphere is always  $R$ , whereas for  $AdS_D$  it is  $(D-3)R/2$ .

We note for completeness that brane solutions which are asymptotically flat can be obtained by replacing  $H_0$  in (4.1)–(4.3) by  $H = 1 + H_0$ . Then, in this context, the length parameter  $R$  has a microscopic interpretation using the eleven-dimensional Planck scale  $l_P$  or the string scale  $\sqrt{\alpha'}$  and the string coupling  $g_s$ , and the number of branes  $N_b$ , which is assumed large. For M5-branes we have  $R^3 = \pi N_b l_P^3$ , for M2-branes  $R^6 = 32\pi N_b l_P^6$ , whereas for D3-branes we have  $R^4 = 4\pi N_b g_s \alpha'^2$ .

## 5 Dualities, smearing and $p$ -brane distributions

In this section we develop a smearing procedure which allows to construct brane distributions for all  $p$ -branes of the type-II string theory. Starting with the M-theory branes we immediately obtain solutions corresponding to distributions of fundamental strings NS1 and D4-branes of type-IIA by simply dimensionally reducing the M2- and M5-brane solutions, respectively, along one of the  $x^\mu$ -coordinates. An S-duality transformation gives

from the NS1 configurations (within its type-IIB interpretation) a solution representing a distribution of D1-branes. The solution representing a distribution of fundamental NS1 strings is given, in the string frame, by

$$\begin{aligned} ds^2 &= H^{-1}(-dt^2 + dx_1^2) + dy_1^2 + \dots dy_8^2, \\ B_{01} &= H^{-1}, \quad e^{-2\Phi} = H. \end{aligned} \quad (5.1)$$

The harmonic function  $H$  is exactly the same as for the M2-branes and the corresponding curves and symmetry groups are given as in table 1. For D1- and D4-branes the corresponding metrics (in the string frame) and dilaton fields, omitting the associated antisymmetric tensor which is also turned on, are given by

$$\begin{aligned} ds^2 &= H^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}(dy_1^2 + dy_2^2 + \dots + dy_{9-p}^2), \\ e^{4\Phi} &= H^{3-p}, \end{aligned} \quad (5.2)$$

with  $p = 1$  and  $p = 4$ , respectively. Again, the harmonic function  $H$  is exactly the same as for the M2- and M5-branes, respectively, and the corresponding curves and symmetry groups are given as in tables 1 and 3.

However, in order to dimensionally reduce along a direction which is transverse to the M-branes, we have to employ a smearing procedure. Recall first that for single-centered solutions, the smearing procedure amounts to simply (re)distributing the branes along an infinite line identified with one of the transverse directions. In this way, the relevant harmonic function becomes independent of the corresponding coordinate, which in turn allows to perform the dimensional reduction. However, in our case this procedure is not applicable as it stands. Instead, we consider the following limit,

$$b_N = -\epsilon^{-2N} g^{3-D}, \quad x^\mu \rightarrow x^\mu \epsilon^{\frac{N}{2(D-2)}}, \quad g \rightarrow g\epsilon, \quad y_i \rightarrow y_i \epsilon^{\frac{D-5}{2}}, \quad (5.3)$$

where  $\epsilon \rightarrow 0$  is an auxiliary dimensionless infinitesimal quantity. We note the explicit appearance of the factor  $g^{3-D}$  in the expression for  $b_N$  above inserted on purely dimensional grounds, since the constants  $b_i$  have dimension of  $(\text{length})^{D-3}$ . This limit is well defined; one easily sees from (4.6) that  $\hat{x}_N = \mathcal{O}(\epsilon^N)$ , whereas the rest of the unit vectors stay finite and define an  $(N-1)$ -sphere. It follows that, in this limit, the dependence on the coordinate  $y_N$  disappears and hence we may consider the dimensional reduction of the M-theory brane solutions (4.1) and (4.2) as before. It is worth emphasizing that the smearing based on (5.3) holds true for all three cases of section 4 (and only for these).

Starting first from the M2-brane solution, we obtain after dimensional reduction along the direction transverse to the brane a solution representing a distribution of D2-branes. The metric and dilaton are given by (5.2) with  $p = 2$ , where the harmonic function  $H$  is given by (4.4) and (4.5) with  $N = 7$  and  $D = 4$ , whereas the corresponding curves and symmetry groups are given in table 4 below.

<i>Genus</i>	<i>Irreducible Curve</i>	<i>Isometry Group</i>
9	$y^4 = (x - b_1)(x - b_2) \cdots (x - b_6)(x - b_7)$	None
7	$y^4 = (x - b_1)(x - b_2) \cdots (x - b_5)(x - b_6)^2$	$SO(2)$
6	$y^4 = (x - b_1)(x - b_2) \cdots (x - b_4)(x - b_5)^3$	$SO(3)$
5	$y^4 = (x - b_1) \cdots (x - b_3)(x - b_4)^2(x - b_5)^2$	$SO(2) \times SO(2)$
4	$y^4 = (x - b_1)(x - b_2)(x - b_3)^2(x - b_4)^3$	$SO(2) \times SO(3)$
3	$y^4 = (x - b_1) \cdots (x - b_3)(x - b_4)^4$	$SO(4)$
	$y^4 = (x - b_1)(x - b_2)(x - b_3)^5$	$SO(5)$
	$y^4 = (x - b_1)(x - b_2)^3(x - b_3)^3$	$SO(3) \times SO(3)$
	$y^4 = (x - b_1)(x - b_2)^2(x - b_3)^2(x - b_4)^2$	$SO(2)^3$
2	$y^4 = (x - b_1)^2(x - b_2)^2(x - b_3)^3$	$SO(2) \times SO(2) \times SO(3)$
1	$y^4 = (x - b_1)(x - b_2)^6$	$SO(6)$
	$y^4 = (x - b_1)(x - b_2)^2(x - b_3)^4$	$SO(2) \times SO(4)$
	$y^4 = (x - b_1)^2(x - b_2)^5$	$SO(2) \times SO(5)$
0	$y^4 = (x - b_1)^3(x - b_2)^4$	$SO(3) \times SO(4)$
	$y^4 = (x - b)^7$	$SO(7)$ (Maximal)

Table 4: Curves and symmetry groups of D2-brane distributions.

After the limiting procedure (5.3) the dimensional reduction of the M5-brane solution along a transverse direction gives a solution representing a distribution of NS5-branes in type-IIA string theory. The string frame metric, the antisymmetric tensor field strength and dilaton field are

$$\begin{aligned}
ds^2 &= -dt^2 + dx_1^2 + \dots + dx_5^2 + H(dy_1^2 + \dots dy_4^2) , \\
e^{2\Phi} &= H , \quad H_{ijk} = \epsilon_{ijkl} \partial_l H , \quad i = 1, 2, 3, 4 ,
\end{aligned} \tag{5.4}$$

where the harmonic function is given by (4.4) and (4.5) with  $N = 4$  and  $D = 7$ . After a T-duality along a direction parallel to the NS5-branes, so that a solution of type-IIB emerges, we can apply an S-duality transformation in order to obtain a solution representing a distribution of D5-branes with metric and dilaton given by (5.2) with  $p = 5$ . For both NS5 and D5 cases, the curves and symmetry groups are given in table 5 below.

<i>Genus</i>	<i>Irreducible Curve</i>	<i>Isometry Group</i>
3	$y^4 = (x - b_1)(x - b_2)(x - b_3)(x - b_4)$	None
1	$y^4 = (x - b_1)(x - b_2)(x - b_3)^2$	$SO(2)$
0	$y^4 = (x - b_1)(x - b_2)^3$	$SO(3)$
	$y^2 = (x - b_1)(x - b_2)$	$SO(2) \times SO(2)$
	$y = (x - b)$	$SO(4)$ (maximal)

Table 5: Curves and symmetry groups of NS5- and D5-brane distributions.

Finally, we may apply a similar smearing procedure to the D5-brane distributions of table 5. After a T-duality transformation, we obtain solutions representing distributions of D6-branes. The harmonic function is given again by (4.4) and (4.5) with  $N = 3$  and  $D = 7$ . The corresponding curves and symmetry groups are given in table 6.

<i>Genus</i>	<i>Irreducible Curve</i>	<i>Isometry Group</i>
3	$y^4 = (x - b_1)(x - b_2)(x - b_3)$	None
1	$y^4 = (x - b_1)(x - b_2)^2$	$SO(2)$
0	$y^4 = (x - b_1)^3$	$SO(3)$ (maximal)

Table 6: Curves and symmetry groups of D6-brane distributions.

In this section we considered so far the limiting procedure in terms of M or string theory solutions. Similar considerations can be made in terms of lower dimensional theories of gauged supergravity, as in [4] for reductions on certain singular limits of  $S^4$ .

## 6 An example of a distribution of NS5-branes

In the case of NS5-branes (or D5-branes) it is possible to explicitly solve the quartic equation (4.5) (with  $N = 4$  and  $D = 7$ ) for  $F$  and substitute the result back into (4.4) in order to obtain an explicit expression for the corresponding harmonic function. However, the resulting expression is not very illuminating to present in detail for general values of the moduli.

We focus attention to distributions of NS5-branes, where the constant  $R$  (in analogy to previous cases) has a microscopic interpretation in terms of the number  $N$  of NS5-branes and the string scale  $\alpha'$  as  $R^2 = N\alpha'$ . For genus zero, besides the solution with isometry  $SO(4)$ , we may explicitly present the solution (5.4). Recall that the solution with symmetry  $SO(2) \times SO(2)$  has already been given in [5] and it represents the field of a large number of NS5-branes uniformly distributed on a circle. In that case it was shown that there is an *exact* conformal field theory corresponding to the coset model  $SL(2, \mathbb{R})/\mathbb{R} \times SU(2)/U(1)$  with level equal to the number  $N$  of NS5-branes.

Here, we present for completeness the other case corresponding to the genus zero surface that preserves an  $SO(3)$  symmetry, according to table 5. For this, it is convenient to use a basis for the unit vectors  $\hat{x}_i$  that define the three-sphere in such a way that it is in one to one correspondence with the decomposition of the vector representation  $\mathbf{4}$  of  $SO(4)$  with respect to the subgroup  $SO(3)$ , as  $\mathbf{4} \rightarrow \mathbf{3} \oplus \mathbf{1}$ . Hence, we choose

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \cos \theta \sin \omega \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix}, \quad \hat{x}_3 = \cos \theta \cos \omega, \quad \hat{x}_4 = \sin \theta. \quad (6.1)$$

It is also convenient to choose the constants  $b_i$  as follows

$$b_1 = b_2 = b_3 = 0, \quad b_4 = -l^2, \quad (6.2)$$

where  $l$  is a real constant. The expressions following (5.4) for the four-dimensional transverse part of the metric, the antisymmetric tensor and the dilaton fields are given explicitly by

$$\begin{aligned} ds^2 &= \frac{1}{4} \left( 1 + \frac{l^2}{r^2} \right)^{1/2} \left( \frac{dr^2}{r^2 + l^2} + d\theta^2 + \frac{r^2 \cos^2 \theta}{r^2 + l^2 \cos^2 \theta} (d\omega^2 + \sin^2 \omega d\varphi^2) \right), \\ B_{\omega\varphi} &= \frac{1}{4} \sin \omega \left( \theta + \frac{r^2 \cos \theta \sin \theta}{r^2 + l^2 \cos^2 \theta} \right), \quad e^{2\Phi} = \frac{\left( 1 + \frac{l^2}{r^2} \right)^{1/2}}{r^2 + l^2 \cos^2 \theta}. \end{aligned} \quad (6.3)$$

In this case, the distribution of NS5-branes is taken over a segment of length  $2l$  along the  $x_4$ -axis with center at the origin. The location of the brane distribution manifests as a naked curvature singularity at  $r = 0$  of the metric in (6.3). Note also that the analytic continuation  $l^2 \rightarrow -l^2$  yields a naked singularity at  $r = l$  corresponding to a distribution of NS5-branes on a three-sphere.

It will be interesting to know whether there is an exact conformal field theory corresponding to the background (6.3).

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